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Abstract: In this work, we study a global output synchronization problem for nonidentical nonlinear systems having relative degree 2 or higher. The synchronization uses a partial projection of individual subsystems into the Brockett oscillators. Our approach is based on output feedback and uses a higher order sliding mode observer to estimate the states and perturbations of the synchronized nonlinear systems. Simulation results are provided to illustrate the performance of the proposed synchronization scheme.

Keywords: Global synchronization; nonidentical nodes; Brockett oscillator; nonlinear systems; higher order sliding modes.

1. INTRODUCTION

Over the last decade, the synchronization of complex dynamical systems and/or network of systems has attracted a great deal of attention from multidisciplinary research communities due to their pervasive presence in nature, technology and human society [Blekhman (1988); Pikovsky et al. (2003); Strogatz (2003); Osipov et al. (2007)]. Among potential application domains of synchronization, it is worth to mention the smooth operations of microgrid [Efimov et al. (2016); Schiffer et al. (2014)], secure communication [Fradkov et al. (2000); Fradkov and Markov (1997)], formation control [Andrievsky and Tomashevich (2015)], chaos synchronization [Rodriguez et al. (2008, 2009)], genetic oscillators [Efimov (2015); Ahmed et al. (2015)], *etc.*

Significant progress has been made during the past decade in the area of control design for synchronization, consensus or motion coordination, the existing literature is huge and covers wide area of topics [Gazi and Passino (2011); Shamma (2007); Olfati-Saber and Murray (2004); Olfati-Saber et al. (2007); Panteley and Loria (2017)]. Until now, a large number of works are available on the problem of synchronization of networks with identical nodes, particularly when the nodes are linear time-invariant systems [Scardovi and Sepulchre (2009); Olfati-Saber et al. (2007); Tomashevich (2017)]. However, most physical systems are often not identical and frequently they are nonlinear in nature. The behavior of dynamical networks with nonidentical nodes is much more complicated than the identical-node case. Usually, no common equilibrium for all nodes exists even if each isolated system has an equilibrium, the same for other invariant solutions which can be destroyed or created by synchronization protocols.

The study of synchronization of dynamical networks with nonidentical nodes is complicated and very few results have been reported by now [Hill and Zhao (2008)]. In [Sun and Geng (2016)], adaptive synchronization has been proposed for nonidentical linear systems. Several collective properties for coupled nonidentical chaotic systems were respectively discussed in [Osipov et al. (1997); Fradkov and Markov (1997); Rodriguez et al. (2009, 2008); Plotnikov et al. (2016)]. Synchronization for smooth and piecewise smooth nonidentical systems with application to Kuramoto oscillators has been studied in [DeLellis et al. (2015)]. Controlled synchronization for two coupled hybrid FitzHugh-Nagumo systems has been studied in [Plotnikov and Fradkov (2016)]. In [Ahmed et al. (2016d)], the authors have proposed robust synchronization for identical/nonidentical multi-stable systems. However, the systems are assumed to admit a decomposition without cycles (neither homoclinic nor heteroclinic orbits). Recently, the results of [Ahmed et al. (2016d)] have been applied to a multi-stable oscillator model [Ahmed et al. (2016b,c, 2017)].

The goal of this work is to address the issue of synchronization of nonidentical nonlinear systems using output feedback only, and an additional subgoal is to have an oscillatory behavior in the synchronized state. Since many engineering systems have relative degree 2 and higher (*e.g.*, pendulum systems [Davila et al. (2005)], oscillators [Rodriguez et al. (2009)], robot manipulators [Salgado et al. (2014)], DC motor [Khalil (2014)]), the particular focus is put on this class of systems. Studying nonidentical systems in general setting, we will assume that neither an equilibrium for each isolated node nor a synchronization manifold exists, so to synchronize them it is necessary

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to apply a feedback transformation [Khalil (2014)] that projects all subsystems to a common dynamics that can be synchronized next. In this work, the Brockett oscillator model is selected for this purpose. This is motivated by a global synchronization control recently proposed for such systems in [Ahmed et al. (2016c)]. Then higher order sliding mode observer is applied to estimate the unmeasurable states and perturbations using the idea presented in [Fridman et al. (2008)]. In short, the main idea is to compensate the nonlinearities of individual systems followed by a nonlinear injection converting some parts of the systems into the Brockett oscillator form. The only restriction is that the individual systems should have relative degree 2 or higher (see Appendix for the definition), but most of the popular oscillator models satisfy this criteria if the output signal is properly selected.

The rest of the article is organized as follows: Section 2 gives the problem statement followed by the synchronizing control design in Section 3. In Section 4, numerical simulation results are given and finally Section 5 concludes this article. Preliminaries on relative degree and a summary of the result of [Ahmed et al. (2016c)] can be found in the Appendix.

2. PROBLEM STATEMENT

The following family of affine controls nonlinear systems is considered in this work for $i = \overline{1, N}$ with $N > 1$:

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_i(x_i)u_i, \\ y_i &= h_i(x_i), \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}$ ($u_i : \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded and measurable signal) is the input, $y_i \in \mathbb{R}$ is the output; $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $h_i : \mathbb{R}^{n_i} \to \mathbb{R}$ and $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ are sufficiently smooth functions. Denote the augmented state vector of (1) as $x = [x_1^T, \dots, x_N^T]^T \in \mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$, $y = [y_1, \dots, y_N]^T \in \mathbb{R}^N$ as the augmented output, and $u = [u_1, \dots, u_N]^T \in \mathbb{R}^N$ as the augmented input. The relative degree condition (see Appendix) imposed on system (1) is summarized by the following assumptions:

Assumption 1. For all $i = \overline{1, N}$, the systems in (1) have global relative degree $r_i \in [2, n_i]$ and globally defined normal form [Marino and Tomei (1996)].

Under this assumption, for each subsystem in (1) there is a diffeomorphic transformation of coordinates $T_i : \mathbb{R}^n \to \mathbb{R}^n$ such that [Marino and Tomei (1996); Khalil (2014)]:

$$\begin{bmatrix} \eta_i \\ \xi_i \end{bmatrix} = T_i(x_i),$$

where $\xi_i \in \mathbb{R}^{r_i}$ and $\eta_i \in \mathbb{R}^{n_i - r_i}$ are new components of the state, and for all $i = \overline{1, N}$ the i^{th} subsystem of (1) can be represented in the normal form:

$$\dot{\eta}_i = \varphi_i(\eta_i, \xi_i), \tag{2}$$

$$\begin{aligned} \dot{\xi}_i &= A_{r_i}\xi_i + b_{r_i}[\alpha_i(\xi_i) + \beta_i(\xi_i)u_i], \\ y_i &= c_{r_i}\xi_i, \end{aligned} \tag{3}$$

where $\varphi_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $\alpha_i : \mathbb{R}^{n_i} \to \mathbb{R}$ and $\beta_i : \mathbb{R}^{n_i} \to \mathbb{R}$ are smooth functions, β_i is separated from zero, and

$$A_{r_i} = \begin{bmatrix} 0 & 1 & 0 \dots 0 & 0 \\ 0 & 0 & 1 \dots 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \dots 0 & 1 \\ 0 & 0 & 0 \dots 0 & 0 \end{bmatrix}, \; b_{r_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$c_{r_i} = [\,1 \;\, 0 \dots 0\,]$$

are in the canonical form. The subsystem (2) is called the *zero dynamics* of i^{th} subsystem in (1), which we assume to be robustly stable:

Assumption 2. For all $i = \overline{1, N}$, the systems in (2) are input-to-state stable (ISS) with respect to the inputs ξ_i [Angeli and Efimov (2015); Dashkovskiy et al. (2011)].

Concerning the definitions of ISS property used in this work, we will not distinguish ISS with respect to a set in the conventional sense [Dashkovskiy et al. (2011)] or for a multistable system [Angeli and Efimov (2015)], the only property we need here is boundedness of the variables η_i for bounded ξ_i. More detailed analysis of the possible asymptotic behavior in (2) for the latter scenario is presented in [Forni and Angeli (2015)].

Then the synchronization problem consists in finding a control u such that the members of the family (1) follow each other. Since the states of the subsystems in (1) may have different dimensions n_i, a state synchronization error $x_i - x_j$ cannot be defined in general (*i.e.* the states of the subsystems in (1) cannot follow their neighbors), but an output synchronization can be formulated:

Definition 1. The family (1) exhibits a *global output synchronization* if

$$\lim_{t \to \infty} (y_i(t) - y_j(t)) = 0, \quad \forall i, j = \overline{1, N}$$

for any initial conditions $x_i(0) \in \mathbb{R}^{n_i}$, $i = \overline{1, N}$.

Note that under Assumption 1 an additional requirement can be imposed on synchronization of derivatives:

$$\lim_{t \to \infty} \{\dot{y}_i(t) - \dot{y}_j(t)\} = 0 \quad \forall i, j = \overline{1, N}.$$

An output feedback controller has to be designed to achieve the global output synchronization for (1).

3. SYNCHRONIZATION CONTROL DESIGN

The idea of this work is to design a feedback controller that will convert a part of subsystems (3) into the form of the Brockett oscillator [Brockett (2013)] through nonlinearity injection. Then global synchronization results can be easily obtained using the control proposed in [Ahmed et al. (2016c)] (a summary is given in Appendix). However, this controller requires all components of the state vector to be available, which limits its implementation. Therefore, to overcome this difficulty, a high-order sliding-mode observer is used.

To simplify the presentation of the forthcoming synchronization protocol design, let us assume that

$$u_i + d_i = \alpha_i(\xi_i) + \beta_i(\xi_i)u_i,$$

where $d_i \in \mathbb{R}$ is a new disturbance signal in (3) for each $i = \overline{1, N}$ (since β_i is not singular, such a representation always exists).

Assumption 3. For all $i = \overline{1, N}$, the unknown input $d_i : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable for almost all $t \geq 0$, and there is a constant $0 < \nu^+ < +\infty$ such that $\operatorname{ess\,sup}_{t \geq 0} |\dot{d}_i(t)| \leq \nu^+$.

3.1 Observer design

Following [Fridman et al. (2008)] and [Levant (2003)], consider, first, for all $i = \overline{1, N}$ a linear observer for (3):

$$\dot{\zeta}_i = A_{r_i}\zeta_i + b_{r_i}u_i + l_i(y_i - c_{r_i}\zeta_i), \tag{4}$$

where $\zeta_i \in \mathbb{R}^{n_i}$ is an auxiliary variable (an estimate of ξ_i), and $l_i \in \mathbb{R}^{n_i}$ is the observer gain designed such that the matrix $A_{r_i} - l_i c_{r_i}$ is Hurwitz. The estimation error $e_i = \xi_i - \zeta_i$ yields the following differential equation:

$$\dot{e}_i = (A_{r_i} - l_i c_{r_i})e_i + b_{r_i}d_i,$$

and to estimate also the unknown input d_i, we consider an extended error vector:

$$\widetilde{e}_i = [e_i^T \ d_i]^T.$$

Then from the available measurement output signal $\psi_i = c_{r_i}e_i$, we obtain:

$$\dot{\widetilde{e}}_i = A_{r_i+1}\widetilde{e}_i + \widetilde{l}_i\psi_i + b_{r_i+1}\dot{d}_i,$$

where $\widetilde{l}_i = [l_i^T \ 0]^T$. Based on [Levant (2003)], the following high order sliding mode differentiator can be applied to estimate the error $\widetilde{e}_i$:

$$\begin{aligned}
\dot{z}_{i,1} &= \nu_{i,1} = -\lambda_{i,1}|z_{i,1} - \psi_i|^{\frac{r_i}{r_i+1}}\operatorname{sign}(z_{i,1} - \psi_i)\\
&\quad + z_{i,2} + \widetilde{l}_{i,1}\psi_i,\\
\dot{z}_{i,j} &= \nu_{i,j} = -\lambda_{i,j}|z_{i,j} - \nu_{i,j-1}|^{\frac{r_i-j+1}{r_i-j+2}}\operatorname{sign}(z_{i,j} - \nu_{i,j-1})\\
&\quad + z_{i,j+1} + \widetilde{l}_{i,j}\psi_i, \ j = \overline{2, r_i},\\
\dot{z}_{i,r_i+1} &= -\lambda_{i,r_i+1}\operatorname{sign}(z_{i,r_i+1} - \nu_{i,r_i}),
\end{aligned} \tag{5}$$

where $\lambda_i = [\lambda_{i,1} \dots \lambda_{i,r_i+1}]^T \in \mathbb{R}^{r_i+1}$ is the vector of tuning parameters. The solutions of the system (1) equipped by the observer (4), (5) are understood in the Filippov sense [Filippov (2013)]. Denote by

$$\widehat{\xi}_i = \zeta_i + \begin{bmatrix} z_{i,1} \\ \vdots \\ z_{i,r_i} \end{bmatrix},$$

$$\widehat{d}_i = z_{i,r_i+1}$$

the estimates of ξ_i and d_i, respectively, provided by the observers (4) and (5). Then the following result can be easily proven:

Proposition 1. Let assumptions 1 and 3 be satisfied, the matrices $A_{r_i} - l_i c_{r_i}$ be Hurwitz and $\lambda_{i,r_i+1} > \nu^+$ for all $i = \overline{1, N}$. Then there exists $\mathcal{T}_i > 0$ such that for the system in (1) (or in (3)) and the observer in (4), (5) for all $t \geq \mathcal{T}_i$:

$$y_i^{(j)}(t) = \widehat{\xi}_{i,j}(t), \ j = \overline{1, r_i},$$
$$d_i(t) = \widehat{d}_i(t),$$

in addition, the estimation errors $\widehat{\xi}_i(t) - \xi_i(t)$ and $\widehat{d}_i(t) - d_i(t)$ stay bounded for all $t \geq 0$.

Proof: Under the assumptions, the system (1) can be transformed to the form (2), (3). In addition, the disturbance d_i can be introduced with a bounded derivative for almost all instants of time. Next, the result follows Lemma 8 in [Levant (2003)], where the finite-time convergence and boundedness of the estimation error for (5) was proven, while the linear observer (4) serves to decouple the external disturbance d_i and the control u_i, which appear in the same equation. ■

The designed observers (4), (5) use only local input-output information u_i and y_i for each node $i = \overline{1, N}$, thus the proposed estimator is completely decentralized.

3.2 Control design

Once we have the estimates of ξ_i and d_i for all $i = \overline{1, N}$, *i.e.* the estimates for the states and the disturbances of the system (3), we are in position to design the synchronization control law.

The relative degree 2 case First, assume that $r_i = 2$. Then the following synchronizing control law can be proposed for all $i = \overline{1, N}$:

$$\begin{aligned}
u_i = &\underbrace{-\widehat{d}_i}_{\text{part 1}} \underbrace{-\widehat{\xi}_{i,1} - b_i\widehat{\xi}_{i,2}\left(\widehat{\xi}_{i,1}^2 + \widehat{\xi}_{i,2}^2 - 1\right)}_{\text{part 2}}\\
&\underbrace{+ a_i k_i\left(\widehat{\xi}_{i-1,2} - 2\widehat{\xi}_{i,2} + \widehat{\xi}_{i+1,2}\right)}_{\text{part 3}},
\end{aligned} \tag{6}$$

where $a_i > 0$, $b_i > 0$ and $k_i > 0$ are tuning parameters. The control law (6) has three parts: part 1 annihilates the nonlinearity of the original system (d_i is dependent on α_i and β_i); while part 2 injects additional nonlinearities to convert the system (3) into the form of the Brockett oscillator; finally, part 3 guarantees synchronization, since it contains the information of the left and right neighbors in the form of (.3) (given in Appendix). In (6), part 1 and part 2 use only local information about the estimates calculated into the node ($\widehat{\xi}_i$ and $\widehat{d}_i$), and only part 3 is based on signals sent over the network in (1). Thus, the control (6) is also decentralized, as in the observer (4), (5), and just the variables $\widehat{\xi}_{i,2}(t)$ have to be communicated.

Theorem 2. Let assumptions 1, 2 and 3 be satisfied, the matrices $A_{r_i} - l_i c_{r_i}$ be Hurwitz and $\lambda_{i,r_i+1} > \nu^+$ for all $i = \overline{1, N}$. Consider the system (1) with the observers (4), (5) and the synchronizing feedback control (6). If there is an index $1 \leq i \leq N$ such that $2a_i k < b_i$, then all trajectories in the closed-loop system are bounded, and for almost all initial conditions, they converge to the largest invariant set where the restrictions are satisfied for all $i = \overline{1, N}$:

$$y_i^2 + \dot{y}_i^2 = \text{ const } \neq 0, \ (y_i - y_{i+1})^2 + (\dot{y}_i - \dot{y}_{i+1})^2 = \text{ const },$$
$$\dot{y}_{i-1} + \dot{y}_{i+1} = (2 + \frac{b_i}{a_i k}(y_i^2 + \dot{y}_i^2 - 1))\dot{y}_i.$$

The higher relative degree case Now, consider the general case with $r_i \geq 2$, then the parts 2 and 3 of the control (6) form a reference signal $\widehat{\xi}_{i,3}^d$ for the variable $\widehat{\xi}_{i,3}$:

$$\begin{aligned}
\widehat{\xi}_{i,3}^d = &-\widehat{\xi}_{i,1} - b_i\widehat{\xi}_{i,2}\left(\widehat{\xi}_{i,1}^2 + \widehat{\xi}_{i,2}^2 - 1\right)\\
&+ a_i k_i\left(\widehat{\xi}_{i-1,2} - 2\widehat{\xi}_{i,2} + \widehat{\xi}_{i+1,2}\right),
\end{aligned}$$

where the parameters $a_i > 0$, $b_i > 0$ and $k_i > 0$ save their meaning, and next this reference signal has to be propagated over chain of integrators, and the part 1 of the control (6) has to be applied on the last step to annihilate d_i. Instead of the usual backstepping [Krstic et al. (1995)], the command filtered backstepping [Farrell et al. (2008)] has to be applied, since it does not need derivatives of the virtual controls (the requirement on derivatives implies that it is necessary to communicate the derivatives of $\widehat{\xi}_{i,2}$ over the network).

4. SIMULATION RESULTS

To illustrate the results presented in Section 3, let us consider the synchronization of a network of Van der Pol oscillators with $N = 3$ as given by [Kanamaru (2007)]:

$$\begin{aligned} \dot{x}_1^i &= x_2^i, i = \overline{1,3} \\ \dot{x}_2^i &= -x_1^i + \mu_i \left\{ 1 - \left(x_1^i \right)^2 \right\} x_2^i + u_i \\ y_i &= x_1^i \end{aligned} \tag{7}$$

where μ_i is the system parameter and is set to $\mu_1 = 0.1$, $\mu_2 = 0.15$ and $\mu_3 = 0.2$ in simulations. Response of the autonomous Van der pol oscillators can be seen in Fig. 1. Model (7) has been successfully applied in various fields like biomedical engineering [Kaplan et al. (2008)], control [Landau et al. (2008)], electrical networks [Sinha et al. (2016)], environmental monitoring [Ahmed et al. (2016a)] *etc.* Because of its practical importance, global synchronization of model (7) is a very interesting problem. However, it is difficult to prove analytically the global synchronization of model (7) with nonidentical nodes. As an alternative way, it is possible to globally synchronize family (7) by transforming individual oscillators into Brockett form as mentioned in Section 3.

With output y_i, the system has relative degree $r_i = n = 2$. Then family (7) can be written in the form (3) as:

$$\dot{\begin{bmatrix} x_1^i \\ x_2^i \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^i \\ x_1^i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[u_i + \underbrace{-x_1^i + \mu_i \left\{ 1 - \left(x_1^i \right)^2 \right\} x_2^i}_{d_i} \right] \tag{8}$$

Then, for system (8) the design of observers (4) and (5) are straightforward. The parameters of observers used in the simulations are $L = [\,20.2 \;\; 102\,]^T$, $\lambda_1 = 3M$, $\lambda_2 = 1.5M$ and $\lambda_3 = 1.1M$ for $M = 1950$. Once the states and unknown inputs are reconstructed through the observers, then the synchronizing control law (6) design becomes trivial. Parameters of the controllers are $a_1 k_1 = 1, a_2 k_2 = 2, a_3 k_3 = 2.5, b_1 = 10$, $b_2 = 20$ and $b_3 = 30$. With these values of the parameters, the condition in Theorem 2 is satisfied. The evolution of the oscillator states with control (6) can be seen in Fig. 2 where it is clear that the control (6) successfully converted the family of Van der Pol oscillators in a finite-time to a family of Brockett oscillators through nonlinearity injection. Then for the family of Brockett oscillators, the result of synchronization follows from Theorem 2. The oscillators in this case converge to the unit circle in the (x_1^i, x_2^i) - space which is similar to the simulation results of [Ahmed et al. (2016c)] for the case of nonidentical oscillators. The unit circle is inside the set Ω'_∞ where the oscillators are supposed to converge from Theorem 4. This demonstrates the effectiveness of the proposed control.

The evolution of control inputs can be seen in Fig. 3. Although the oscillators are synchronized as seen in Fig. 2, the control signals do not become zero because of the continuous nonlinearity injection as shown in Fig. 3. The evolution of observation errors can be seen in Fig. 4, where the estimation errors converge in finite-time.

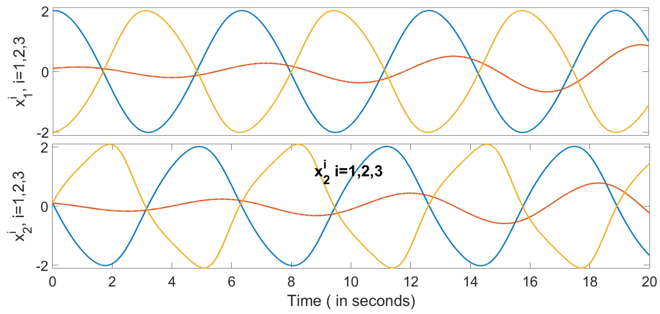

Figure 1. Evolution of the state variables of model (7).

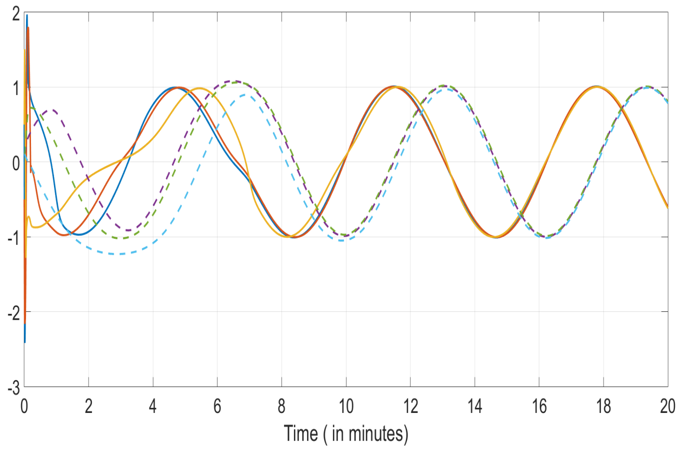

Figure 2. Evolution of the oscillator states. Solid lines - x_{2i} and dashed lines - x_{1i}

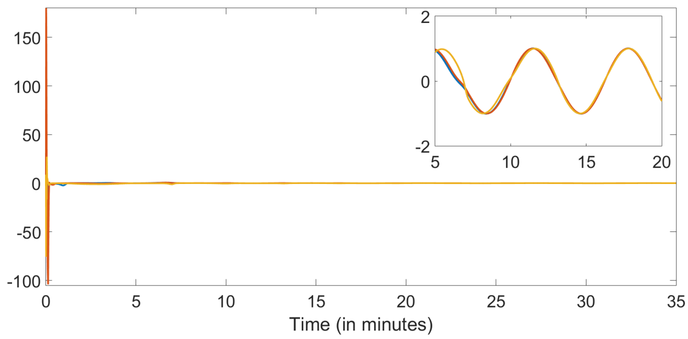

Figure 3. Evolution of control inputs. Zoomed version in the inset.

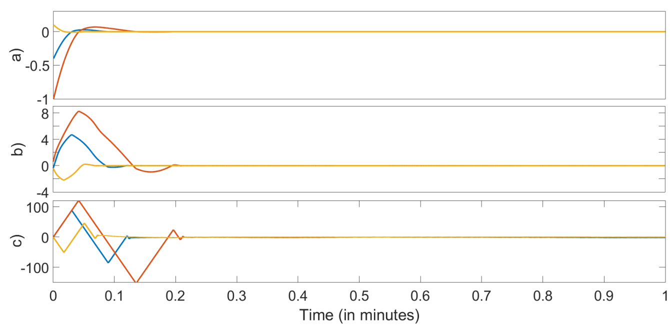

Figure 4. Estimation errors: (a) $x_1^i - \widehat{x}_1^i$, (b) $x_2^i - \widehat{x}_2^i$ and (c) $d_i - \widehat{d}_i$.

5. CONCLUSION

This paper studied the problem of global synchronization of nonlinear systems with relative degree 2 and higher using output feedback. The nonlinear systems were first converted to an normal canonical form. Then higher order sliding mode observers were used to reconstruct the states and the perturbations in a finite time. Using these information, individual systems were projected to Brockett oscillator dynamics through feedback control. Numerical simulations demonstrated the effectiveness of our method using a network of nonidentical Van der Pol oscillators.

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APPENDIX

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output variable of the system, f and g are smooth vector fields. A vector field is said to be *complete* if all solutions to the $\dot{x} = f(x)$ are defined for all $t \geq 0$ [Khalil (2014)].

Definition 2. (Global Uniform Relative Degree [Marino and Tomei (1996)]). The *global uniform relative degree* r of (.1) is defined as the integer such that

$$\begin{aligned} L_g L_f^i h(x) = 0, &\quad \forall x \in \mathbb{R}^n,\ 0 \leq i \leq r-2, \\ L_g L_f^{r-1} h(x) \neq 0, &\quad \forall x \in \mathbb{R}^n. \end{aligned}$$

We say that $r = \infty$ if

$$L_g L_f^i h(x) = 0, \quad \forall x \in \mathbb{R}^n,\ \forall i \geq 0.$$

SYNCHRONIZATION OF BROCKETT OSCILLATORS [AHMED ET AL. (2016C)]

The following family of Brockett oscillators [Brockett (2013)] is considered in this section for some $N > 1$:

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= a_i u_i - x_{1i} - b_i x_{2i}\left(|x_i|^2 - 1\right), i = \overline{1, N}, \end{aligned} \tag{.2}$$

where $a_i, b_i > 0$ are the parameters of an individual oscillator, the state $x_i = [x_{1i}\ \ x_{2i}]^T \in \mathbb{R}^2$ and the control $u_i \in \mathbb{R}$ ($u_i : \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded and measurable signal). Denote the common state vector of (.2) by $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^{2N}$ and the common input by $u = [u_1, \ldots, u_N]^T \in \mathbb{R}^N$.

The following synchronizing control is selected for family (.2):

$$u = kM \begin{bmatrix} x_{21} \\ \vdots \\ x_{2(N-1)} \\ x_{2N} \end{bmatrix}, \tag{.3}$$

where $k > 0$ is the coupling strength and

$$M = \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 1 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

From a graph theory point of view, the oscillators are connected through a N-cycle graph [Pemmaraju and Skiena (2003)] (each oscillator needs only the information of its left and right neighbor). Define the synchronization error among the various states of the oscillators as:

$$e_{2i-1} = x_{1i} - x_{1(i+1)},\ \dot{e}_{2i-1} = x_{2i} - x_{2(i+1)} = e_{2i}$$

and $e_{2N-1} = x_{1N} - x_{11}$, $\dot{e}_{2N-1} = x_{2N} - x_{21} = e_{2N}$. Then the main results of [Ahmed et al. (2016c)] can be summarized as below:

Proposition 3. For any $k > 0$ in the system (.2), (.3) all trajectories are bounded and converge to the largest invariant set in

$$\begin{aligned} \Omega_\infty = \big\{ & x \in M : |x_i| = \text{const},\ e_{2i-1}^2 + e_{2i}^2 = \text{const}, \\ & x_{2(i-1)} + x_{2(i+1)} = (2 + \frac{b_i}{a_i k}(|x_i|^2 - 1))x_{2i}, i = \overline{1, N} \big\}. \end{aligned}$$

Theorem 4. For any $k > 0$, if there is an index $1 \leq i \leq N$ such that $2a_i k < b_i$, then in the system (.2), (.3) all trajectories are bounded and almost all of them converge to the largest invariant set in

$$\begin{aligned} \Omega'_\infty = \big\{ & x \in M : |x_i| = \text{const} \neq 0,\ e_{2i-1}^2 + e_{2i}^2 = \text{const}, \\ & x_{2(i-1)} + x_{2(i+1)} = (2 + \frac{b_i}{a_i k}(|x_i|^2 - 1))x_{2i}, i = \overline{1, N} \big\}. \end{aligned}$$